Summability Classes of Δ^s-Sequences of Interval Numbers

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Abstract: In this article we introduce and study the notions generalized difference lacunary strongly summable, Cesàro strongly summable, - statistically convergent and Δ^s -lacunary statistically convergent sequence of interval numbers. Consequently we construct the sequence classes $\ell_\theta^i\left(\Delta^s\right)$, $\sigma_1^i\left(\Delta^s\right)$, $s^i\left(\Delta^s\right)$ and $s_\theta^i\left(\Delta^s\right)$ respectively and investigate the relationship among these classes.

Keywords: Sequence of interval numbers; Difference sequence; lacunary strongly summable; Cesàro strongly summable; statistically convergent; lacunary statistically convergent; Completeness.

1. Introduction

The concept of interval arithmetic was first suggested by Dwyer [1] in 1951. After developed by Moore [10], Moore and Yang [13]. Furthermore several authors have studied various aspects of the theory and applications of interval numbers in differential equations [13], [14], [15]. The sequence of interval numbers was first introduced by Chiao [20] and defined usual convergence. Bounded and convergence sequences spaces of interval numbers were introduced by Sengonul and Eryilmaz [18] and showed that these spaces are complete metric space.

A set consisting of closed interval of real numbers x such that $a \le x \le b$ is called an interval number. A real interval can also be considered as a set. Denote the set of all real valued closed intervals by $\mathbb{R}$. Any member of $\mathbb{R}$ is called closed interval and denoted by $\bar{x}$. Thus $\bar{x} = \left\{x \in \mathbb{R} : a \le x \le b\right\}$. In [20], an interval number is closed subset of real line $\mathbb{R}$.

Let x_l and x_r be the first and last points of the interval number $\bar{x}$ respectively. For $\bar{x}_1$, $\bar{x}_2 \in \mathbb{R}$, we have

$$\bar{x}_1 = \bar{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}, x_{1r} = x_{2_r}.$$

$$\bar{x}_1 + \bar{x}_2 = \left\{x \in \mathbb{R} : x_{1_l} + x_{2_l} \le x \le x_{1r} + x_{2_r}\right\}$$

$$\alpha\bar{x} = \left\{x \in \mathbb{R} : \alpha x_{1_l} \le x \le \alpha x_{1_r}\right\} \text{ if } \alpha \ge 0.$$

$$= \left\{x \in \mathbb{R} : \alpha x_{1_r} \le x \le \alpha x_{1_l}\right\} \text{ if } \alpha < 0.$$

and

$$\bar{x}_1 . \bar{x}_2 = \left\{x \in \mathbb{R} : \min\left(x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\right) \le x \le \max\left(x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\right)\right\}$$

The set of all interval numbers $\mathbb{R}$ is complete metric space under the metric defined by –

$$d\left(\bar{x}, \bar{y}\right) = \max\left\{\left|x_{1_l} - x_{2_l}\right|, \left|x_{1_r} - x_{2_r}\right|\right\}$$ (see [18]) .

Let us consider the transformation $f:\mathbb{N}\to\mathbb{R}$ by $k\to f(k)=\bar{x}$ where $\bar{x}=(\bar{x}_k)$ which is known as sequence of interval numbers. $\bar{x}_k$ denotes the k^{th} term of the sequence $\bar{x}=(\bar{x}_k)$. The set of all sequences of interval numbers is denoted by w^i can be found in [18].

2. Definitions and Main Results

Throughout the article we denote by w^i the set of all sequences $\bar{x}=(\bar{x}_k)$ of interval numbers.

By a lacunary sequence $\vartheta=(k_p)$; $p = 1, 2, 3 \ldots$, where $k_o = 0$, we mean an increasing sequence of non-negative integers with $h_p = (k_p - k_{p-1}) \to \infty$ as $p \to \infty$. We denote $I_p = (k_{p-1}, k_p]$ for $p = 1, 2, 3 \ldots$

Let r and s be two non-negative integers and $v = (v_k)$ be a sequence of non-zero reals. Then for a lacunary sequence ϑ we define:

$$\ell_\theta^i(\Delta^s)=\left\{\bar{x}=(\bar{x}_k)\in w^i:\lim_{p\to\infty}\frac{1}{h_p}\sum_{k\in I_p}d\left(\Delta^s\bar{x}_k,\bar{x}_0\right)=0,\text{for some }\bar{x}_0\right\},$$

Where $\left(\Delta^s\bar{x}_k\right)=\left(\Delta^{s-1}\bar{x}_k-\Delta^{s-1}\bar{x}_{k-1}\right)$ and $\Delta^0\bar{x}_k=\bar{x}_k$ for all $k\in N$, which is equivalent to the following binomial representation:

$$\Delta^s\bar{x}_k=\sum_{i=0}^{s}(-1)^i\binom{s}{i}v_{k-i}\,\bar{x}_{k-i}.$$

In this expansion it is important to note that we take $v_{k-i}=0$ and $\bar{x}_{k-i}=\bar{0}$ for non-positive values of k-i.

If $\bar{x}\in\ell_\theta^i(\Delta^s)$, then we says that $\bar{x}$ is Δ^s-lacunary strongly summable sequence of interval numbers.

A sequence $\bar{x}=(\bar{x}_k)\in w^i$ is said to be Δ^s- Cesàro strongly summableif $\bar{x}\in\sigma_1^i(\Delta^s)$, where

$$\sigma_1^i(\Delta^s)=\left\{\bar{x}\in w^i:\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}d\left(\Delta^s\bar{x}_k,\bar{x}_0\right)=0,\text{for some }\bar{x}_0\right\}$$

A sequence $\bar{x}=(\bar{x}_k)\in w^i$ is said to be Δ^s- statistically convergent if $\bar{x}\in S^i(\Delta^s)$, where

$$S^i(\Delta^s)=\left\{\bar{x}\in w^i:\lim_{n}\frac{1}{n}\text{card}\left\{k\le n:d(\Delta^s\bar{x}_k,\bar{x}_0)\ge\varepsilon\right\}=0,\text{for every }\varepsilon>0\text{ and some }\bar{x}_0\right\}$$

A sequence $\bar{x}=(\bar{x}_k)\in w^i$ is said to be Δ^s-lacunary statistically convergent if $\bar{x}\in S_\theta^i(\Delta^s)$, where

$$S_\theta^i(\Delta^s)=\left\{\bar{x}\in w^i:\lim_{p}\frac{1}{h_p}\text{card}\left\{k\in I_p:d(\Delta^s\bar{x}_k,\bar{x}_0)\ge\varepsilon\right\}=0,\text{for every }\varepsilon>0\text{ and some }\bar{x}_0\right\}$$

In this case we write $s_\theta-\lim\bar{x}_k=\bar{x}_0$

Theorem 2.1: Let $\bar{x}=(\bar{x}_k)$ and $\bar{y}=(\bar{y}_k)$ be sequences of interval numbers, then-

(1) If $S_\theta^i-\lim\Delta^s\bar{x}_k=\bar{x}_0$ and $\alpha\in R$ then $S_\theta^i-\lim\Delta^s\alpha\bar{x}_k=\alpha\bar{x}_0$.

(2) If $S_\theta^i-\lim\Delta^s\bar{x}_k=\bar{x}_0$ and $S_\theta^i-\lim\Delta^s\bar{y}_k=\bar{y}_0$, then $S_\theta^i-\lim(\Delta^s\bar{x}_k+\Delta^s\bar{y}_k)=\bar{x}_0+\bar{y}_0$

Proof:

(1) Let $\alpha \in R$. We have, $d(\Delta^s \alpha \bar{x}_k, \alpha \bar{x}_0) = |\alpha| d(\Delta^s \bar{x}_k, \bar{x}_0)$.

For any $\varepsilon > 0$,

$$\frac{1}{h_p} \text{card}\left\{k \in I_p : d(\Delta^s \alpha \bar{x}_k, \alpha \bar{x}_0) \geq \varepsilon\right\} \leq \frac{1}{h_p} \text{card}\left\{k \in I_p : d(\Delta^s \bar{x}_k, \bar{x}_0) \geq \frac{\varepsilon}{|\alpha|}\right\}$$ Hence

$$s_\theta - \lim \Delta^s_{(v,r)} \alpha \bar{x}_k = \alpha \bar{x}_0.$$

(2) Suppose $S_\theta^i - \lim \Delta^s \bar{x}_k = \bar{x}_0$ and $S_\theta^i - \lim \Delta^s \bar{y}_k = \bar{y}_0$

We have,

$$d(\Delta^s \bar{x}_k + \Delta^s \bar{y}_k, \bar{x}_0 + \bar{y}_0) \leq d(\Delta^s \bar{x}_k, \bar{x}_0) + d(\Delta^s \bar{y}_k, \bar{y}_0)$$

So, for any given $\varepsilon > 0$,

$$\frac{1}{h_p} \text{card}\left\{k \in I_p : d(\Delta^s \bar{x}_k + \Delta^s \bar{y}_k, \bar{x}_0 + \bar{y}_0) \geq \varepsilon\right\}$$

$$\leq \frac{1}{h_p} \text{card} \left\{k \in I_p : d(\Delta^s \bar{x}_k, \bar{x}_0) + d(\Delta^s \bar{y}_k, \bar{y}_0) \geq \varepsilon\right\}$$

$$\leq \frac{1}{h_p} \text{card} \left\{k \in I_p : d(\Delta^s \bar{x}_k, \bar{x}_0) \geq \frac{\varepsilon}{2}\right\}$$

$$+ \frac{1}{h_p} card \left\{k \in I_p : d(\Delta^s \bar{y}_k, \bar{y}_0) \geq \frac{\varepsilon}{2}\right\}$$

Thus, $s_\theta^i - \lim(\Delta^s \bar{x}_k + \Delta^s \bar{y}_k) = \bar{x}_0 + \bar{y}_0$.

Theorem 2.2. Let θ be a lacunary sequence. Then if a sequence $\bar{x} = \left(\bar{x_k}\right)$ of interval numbers is Δ^s -lacunary strongly summable then it is Δ^s - lacunary statistically convergent.

Proof. Suppose $\bar{x} = \left(\bar{x_k}\right)$ is strongly Δ^s -lacunary strongly summable to X_0. Then

$$\lim_{p\to\infty} \frac{1}{h_p} \sum_{k \in I_p} d\left(\Delta^s \bar{x}_k, \bar{x}_0\right) = 0.$$

Now the result follows from the following inequality:

$$\sum_{k \in I_p} d\left(\Delta^s \bar{x}_k, \bar{x}_0\right) \geq \varepsilon \text{card}\left\{k \leq n : d(\Delta^s \bar{x}_k, \bar{x}_0) \geq \varepsilon\right\}$$

Theorem 2.3. If a sequence $\bar{x} = \left(\bar{x_k}\right)$ of interval numbers is Δ^s -bounded and Δ^s - statistically convergent, then it is Δ^s - Cesàro strongly summable.

Proof. Suppose $\bar{x} = \left(\bar{x_k}\right)$ is Δ^s -bounded and Δ^s - statistically convergent to $\bar{x}_0$. Since $\bar{x} = \left(\bar{x_k}\right)$ is Δ^s -bounded, we can find a interval number *M* such that

$d(\Delta^s \bar{x}_k, \bar{x}_0) \leq M$ for all $k \in N$

Again since $\bar{x} = \left(\bar{x_k}\right)$ is Δ^s - statistically convergent to $\bar{x}_0$, for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} \text{card}\left\{k \leq n : d(\Delta^s \bar{x}_k, \bar{x}_0) \geq \varepsilon\right\} = 0,$$

Now the result follows from the following inequality:

$$\frac{1}{n}\sum_{1\le k\le n} d\left(\Delta^s \bar{x}_k, \bar{x}_0\right) = \frac{1}{n}\sum_{\substack{1\le k\le n\\ d(\Delta^s_{(v,r)}\bar{x}_k,\bar{x}_0)\ge\varepsilon}} d(\Delta^s \bar{x}_k, \bar{x}_0) + \frac{1}{n}\sum_{\substack{1\le k\le n\\ d(\Delta^s_{(v,r)}\bar{x}_k,\bar{x}_0)<\varepsilon}} d(\Delta^s \bar{x}_k, \bar{x}_0)$$

$$\le \frac{M}{n}\operatorname{card}\left\{k\le n : d(\Delta^s \bar{x}_k, \bar{x}_0)\ge\varepsilon\right\} + \varepsilon$$

Theorem 2.4. Let θ be a lacunary sequence. Then if a sequence $\bar{x} = \left(\overline{x_k}\right)$ is Δ^s -bounded and Δ^s - lacunary statistically convergent, then it is Δ^s - lacunary strongly summable.

Proof. Proof follows by similar arguments as applied to prove above Theorem.

Theorem 2.5. Let θ be a lacunary sequence and $\bar{x} = \left(\overline{x_k}\right)$ be Δ^s -bounded. Then X is Δ^s - lacunary statistically convergent if and only if it is Δ^s - lacunary strongly summable.
Proof. Proof follows by combining the Theorems 2.1 and 2.3.

Theorem 2.6. If a sequence $\bar{x} = \left(\overline{x_k}\right)$ is Δ^s - statistically convergent and liminf$_p\left(\frac{h_p}{p}\right)$ > 0 then it is Δ^s -lacunary statistically convergent.

Proof. Assume the given conditions. For a given ε> 0, we have

$$\left\{k\in I_p : d(\Delta^s \bar{x}_k, \bar{x}_0)\ge\varepsilon\right\} \subset \left\{k\le n : d(\Delta^s \bar{x}_k, \bar{x}_0)\ge\varepsilon\right\}$$

Hence the proof follows from the following inequality:

$$\frac{1}{p}\operatorname{card}\left\{k\le p : d(\Delta^s \bar{x}_k, \bar{x}_0)\ge\varepsilon\right\} \ge \frac{1}{p}\operatorname{card}\left\{k\in I_p : d(\Delta^s \bar{x}_k, \bar{x}_0)\ge\varepsilon\right\}$$

$$= \frac{h_p}{p}\frac{1}{h_p}\operatorname{card}\left\{k\in I_p : d(\Delta^s \bar{x}_k, \bar{x}_0)\ge\varepsilon\right\}$$

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